Phase conjugation of continuous quantum variables

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(December 2000)

The phase conjugation of an unknown Gaussian state cannot be realized perfectly by any physical process. A semi-classical argument is used to derive a tight lower bound on the noise that must be introduced by an approximate phase conjugation operation. A universal transformation achieving the optimal imperfect phase conjugation is then presented, which is the continuous counterpart of the universal-NOT transformation for quantum bits. As a consequence, it is also shown that more information can be encoded into a pair of conjugate Gaussian states than using twice the same state.

PACS numbers: 03.67.-a, 03.65.Bz, 42.50.Dv

arXiv:quant-ph/0012020v1 5 Dec 2000

The *spin-flip* operation cannot be performed on an arbitrary spin-1/2 particle (or qubit) since it is an *anti-unitary* transformation. In other words, given a spin-1/2 particle polarized in an unknown direction $\vec{n}$, the state $|\vec{n}\rangle$ cannot be turned into $|-\vec{n}\rangle$ by any physical process. Recently, however, it has been shown that this operation can be done *imperfectly*, with a same fidelity for all states $|\vec{n}\rangle$, by using a universal quantum spin-flip (or universal-NOT) transformation [1,2]. This transformation yields $|-\vec{n}\rangle$ with a fidelity of 2/3, which, remarkably, coincides with the fidelity of the optimal measurement of a spin-1/2 particle [3]. This means that the optimal spin-flip operation can be achieved by first measuring the spin in an arbitrary direction, then preparing a spin state pointing in the opposite direction to the measured spin. A related result is that encoding a space direction $\vec{n}$ into two antiparallel spins $|\vec{n},-\vec{n}\rangle$ is slightly more efficient than using a naive encoding with parallel spins $|\vec{n},\vec{n}\rangle$ [2].

In this paper, we investigate the continuous analogue of the spin-flip operation, namely the *phase conjugation* (or, equivalently, time reversal). First, we analyze the impossibility of perfectly conjugating an arbitrary Gaussian state (or, in particular, a coherent state $|\alpha\rangle$). We find that such a process necessarily effects a noise that is equal to at least *twice* the vacuum fluctuation noise of the input coherent state. This leads us to define a universal phase conjugator or universal-NOT operator for continuous quantum variables. We then show that this transformation is optimal as it achieves the lower bound derived above. The resulting phase conjugation fidelity is 1/2, which, just as for qubits, is the same as the fidelity of the optimal measurement of a coherent state [4–6]. Finally, the link with quantum state estimation and quantum cloning for coherent states is discussed. In particular, it is shown that, in analogy with the situation for qubits, it is more efficient to encode information into a pair of conjugate coherent states $|\alpha\rangle\otimes|\alpha^*\rangle$ rather than using twice the same state $|\alpha\rangle^{\otimes 2}$. The error variance on the real and imaginary parts of α can actually be divided by *two* in the former case (by applying an appropriate *entangled* measurement), with respect to the latter case.

Consider a single mode of the electromagnetic field, denoted as $\hat{a}=(\hat{x}+i\hat{p})/\sqrt{2}$. The phase conjugation operation consists in flipping the sign of quadrature $\hat{p}$ while keeping quadrature $\hat{x}$ unchanged, that is, replacing $\hat{a}$ by its Hermitian conjugate $\hat{a}^\dagger$. Clearly, this operation is impossible as it does not conserve the commutation relation: if $\hat{b}=\hat{a}^\dagger$ is the resulting mode, we have $[\hat{b},\hat{b}^\dagger]=-[\hat{a},\hat{a}^\dagger]=-1$ instead of 1 ($\hbar=1$). A semi-classical argument can be used to show that this operation cannot be performed with an added noise that is lower than a minimum equal to twice the vacuum noise. Let us consider two modes (mode 0 and 1) that are initially prepared in the Einstein-Podolsky-Rosen (EPR) state, that is, the common eigenstate of $\hat{X}=\hat{x}_0-\hat{x}_1$ and $\hat{P}=\hat{p}_0+\hat{p}_1$ with zero eigenvalue for both operators $\hat{X}$ and $\hat{P}$. Since $[\hat{X},\hat{P}]=0$, these operators can be diagonalized simultaneously, so that the EPR state can be understood as representing two particles with a relative position x_0-x_1 and a total momentum p_0+p_1 both arbitrarily close to zero. Assume now that we apply a phase conjugation operator on mode 1, that is, $\hat{x}_1'=\hat{x}_1$ and $\hat{p}_1'=-\hat{p}_1$, while mode 0 is left unchanged. The EPR state is then transformed into the common eigenstate with zero eigenvalue of operators $\hat{X}'$ and $\hat{P}'$, defined as

$$\begin{aligned}\hat{x}_0-\hat{x}_1&=\hat{x}_0'-\hat{x}_1'\equiv\hat{X}',\\ \hat{p}_0+\hat{p}_1&=\hat{p}_0'-\hat{p}_1'\equiv\hat{P}'.\end{aligned}\tag{1}$$

Importantly, $\hat{X}'$ and $\hat{P}'$ cannot commute any more here if the transformed modes 0' and 1' are to obey the standard commutation relations, so it is indeed impossible to obtain a common eigenstate of $\hat{x}_0'-\hat{x}_1'$ and $\hat{p}_0'-\hat{p}_1'$. Instead, since $[\hat{X}',\hat{P}']=[\hat{x}_0',\hat{p}_0']+[\hat{x}_1',\hat{p}_1']=2i$, the Heisenberg uncertainty relation implies that

$$\Delta\hat{X}'\,\Delta\hat{P}'\geq\frac{1}{2}|\langle[\hat{X}',\hat{P}']\rangle|=1.\tag{2}$$

If we now assume that the phase conjugation process introduces some noise, then it is easy to determine the

minimum amount of such noise for the Heisenberg uncertainty relation to be satisfied. Let us suppose that mode 1 suffers, after phase conjugation, from a random noise n_x and n_p on quadrature $\hat{x}'_1$ and $\hat{p}'_1$, respectively. Thus, $\hat{x}'_1 = \hat{x}_1 + n_x$ and $\hat{p}'_1 = -\hat{p}_1 + n_p$. Naturally, we assume that this noise is unbiased, that is, $\langle n_x \rangle = \langle n_p \rangle = 0$. Since we are seeking for a "universal" transformation, we require the variances of n_x and n_p to be the same ($\langle n_x^2 \rangle = \langle n_p^2 \rangle = \sigma^2$). The resulting variance of operators $\hat{X}' = \hat{x}_0 - \hat{x}_1 - n_x$ and $\hat{P}' = \hat{p}_0 + \hat{p}_1 - n_p$ is

$$\Delta \hat{X}'^2 = \Delta \hat{P}'^2 = \sigma^2, \tag{3}$$

since $\hat{x}_0 - \hat{x}_1$ and $\hat{p}_0 + \hat{p}_1$ have both a vanishing variance in the EPR state. Equation (2) then implies that

$$\sigma^2 \geq 1, \tag{4}$$

so that the noise induced by the phase conjugation process is lower bounded by 1, i.e., *twice* the variance of a quadrature in the vacuum state ($\Delta x^2_{\rm vac} = 1/2$).

Let us now construct an actual phase-conjugating transformation that attains this bound. The input mode, assumed to be prepared in a coherent state $|\alpha\rangle$, is coupled to an ancilla mode by some unitary transformation. Subsequently, the ancilla is traced over, so the processed mode is left in a mixed state that is required to be as close as possible to the complex conjugate state $|\alpha^*\rangle$. Let us denote the input mode by $\hat{a}_1$ and the ancilla mode by $\hat{a}_2$. The canonical transformation can be generally described as

$$\hat{b}_i = M_{ij} \hat{a}_j + L_{ij} \hat{a}_j^\dagger, \tag{5}$$

where $i, j = 1, 2$, and the sum is implicit. The output modes $\hat{b}_1$ and $\hat{b}_2$ refer to the phase-conjugator output and the processed ancilla, respectively. This transformation is determined, in general, by 8 complex coefficients, but we will now impose the constraints for it to characterize an (imperfect) phase conjugator. First, we note that it is always possible to perform a phase transformation $\hat{a}_i \to e^{i\phi_i} \hat{a}_i$ and $\hat{b}_i \to e^{i\psi_i} \hat{b}_i$ such that the coefficients M_{1j} and L_{1j} are real and positive. Then, by definition, we require that the phase conjugator obeys $\langle \hat{b}_1 \rangle = \langle \hat{a}_1^\dagger \rangle$. Also, without loss of generality, we can assume that the ancilla is initially in the vacuum state $\langle \hat{a}_2 \rangle = \langle (\hat{a}_2)^2 \rangle = 0$ (see [7]). Thus, we must have $M_{11} = 0$ and $L_{11} = 1$. We now impose the "universality" of the transformation, that is, the constraint that the added noise is phase-insensitive (each quadrature suffers from a the same noise). If the input mode has phase-insensitive noise, i.e., if $\langle (\hat{a}_1)^2 \rangle = \langle \hat{a}_1 \rangle^2$ (for example, if it is a coherent state), then we require that the output mode also has phase-insensitive noise, i.e., $\langle (\hat{b}_1)^2 \rangle = \langle \hat{b}_1 \rangle^2$. Using

$$\langle (\hat{b}_1)^2 \rangle - \langle \hat{b}_1 \rangle^2 = \langle (\hat{a}_1^\dagger)^2 \rangle - \langle \hat{a}_1^\dagger \rangle^2 + M_{12} L_{12} \tag{6}$$

we conclude that the universality condition is simply $M_{12} L_{12} = 0$. Three more conditions come from imposing the commutation rules to be conserved by the transformation (5):

$$[b_1, b_1^\dagger] = M_{12}^2 - L_{12}^2 - 1 = 1, \tag{7}$$

$$[b_2, b_2^\dagger] = |M_{21}|^2 + |M_{22}|^2 - |L_{21}|^2 - |L_{22}|^2 = 1, \tag{8}$$

$$[b_1, b_2] = M_{1j} L_{2j} - L_{1j} M_{2j} = 0. \tag{9}$$

Equation (7), together with the universality condition, implies that $L_{12} = 0$ and $M_{12} = \sqrt{2}$. Equations (8) and (9) then impose two last conditions on the four coefficients M_{2j} and L_{2j}, so we are left with two free parameters. If we further impose that mode 2 transforms just as mode 1 ($M_{22} = 0$ and $L_{22} = 1$), then we get

$$\hat{b}_1 = \hat{a}_1^\dagger + \sqrt{2}\, \hat{a}_2, \tag{10}$$

$$\hat{b}_2 = \sqrt{2}\, \hat{a}_1 + \hat{a}_2^\dagger. \tag{11}$$

As we could expect, this transformation exactly describes a phase-insensitive phase-conjugating linear amplifier (see [7]). One can easily check that the noise variance of the output of this phase conjugator is

$$(\Delta x^2)_{b_1} = (\Delta p^2)_{b_1} = \Delta x^2_{\rm vac} + 2\Delta x^2_{\rm vac} = 3/2 \tag{12}$$

so that the phase-conjugation induced noise is *twice* the vacuum noise, i.e., $2\Delta x^2_{\rm vac} = 1$. Hence, this transformation is optimal as it saturates the bound (4). In particular, if the input is a coherent state $|\alpha\rangle$, the output will be a Gaussian mixture of coherent state ρ with variance one centered on $|\alpha^*\rangle$. Consequently, the phase conjugating fidelity is

$$F = \langle \alpha^* | \rho | \alpha^* \rangle = 1/2, \tag{13}$$

just as for an optimal measurement [4–6]. Interestingly, this implies that phase conjugation is intrinsically a classical process. It could be achieved as well by simultaneously measuring the two quadratures of $|\alpha\rangle$, and then preparing a coherent state whose quadrature p has a flipped sign. Incidentally, we note that any number of phase-conjugated outputs can actually be prepared together at no cost (with $F = 1/2$ for each).

It is interesting, at this point, to extend the parallel with the universal quantum spin-flip machine for qubits, and make a connection with a state estimation question. In [2], Gisin and Popescu have found the surprising result that encoding a direction $\vec{n}$ into two antiparallel spins $|\vec{n}, -\vec{n}\rangle$ yields slightly more information on $\vec{n}$ than encoding it into two parallel spins $|\vec{n}, \vec{n}\rangle$. Here, we investigate the counterpart of this situation for information that is carried by a continuous quantum variable instead of a qubit. Consider the situation where Alice wants to communicate to Bob a complex number $\alpha = (x + ip)/\sqrt{2}$. Assume Alice is allowed to use a quantum channel only twice so as to send Bob two coherent states of a given

amplitude $|\alpha|^2$ each. She can choose, for example, to send Bob the product state $|\alpha\rangle^{\otimes 2}$. In this case, the best strategy to infer both x and p with a same precision is to perform a product measurement [5]. A simultaneous measurement of the two quadratures of each coherent state $|\alpha\rangle$ yields (x,p) with a variance $2\Delta x^2_{\rm vac} = 1$ [4]. The resulting error variance on x and p estimated from these two measurements is then equal to one half of this variance, that is $\Delta x^2_{\rm vac} = 1/2$. (This is just the statistical factor.)

Another possibility is that Alice sends Bob the product state $|\alpha\rangle \otimes |\alpha^*\rangle$. In this case, a possible (but not necessarily optimal) strategy for Bob is again to carry out a product measurement, taking into account that the measured value of p of the second state should be read as $-p$. This obviously results in the same error variance 1/2. However, the fact that the continuous universal-NOT transformation has a non-unity fidelity leaves open the possibility that there exists a measurement of $|\alpha\rangle \otimes |\alpha^*\rangle$ that is *not* of a product form, and yields a variance strictly lower than 1/2. Indeed, if there was a perfect universal phase conjugator, then it could be used to convert $|\alpha^*\rangle$ into $|\alpha\rangle$ before applying the optimal product measurement on $|\alpha\rangle^{\otimes 2}$, thereby resulting in the same minimum variance in both cases.

Let us now explicitly describe an *entangled* measurement of the product state $|\alpha\rangle \otimes |\alpha^*\rangle$, which yields indeed a lower variance. Expressing the two input modes as $|\alpha\rangle = \exp(ip\hat{x}_1 - ix\hat{p}_1)|0\rangle$ and $|\alpha^*\rangle = \exp(-ip\hat{x}_2 - ix\hat{p}_2)|0\rangle$, we can write the input product state as $|\alpha\rangle \otimes |\alpha^*\rangle = \exp(ip\hat{X} - ix\hat{P})|0\rangle$, where $\hat{X} = \hat{x}_1 - \hat{x}_2$ and $\hat{P} = \hat{p}_1 + \hat{p}_2$ are two *commuting* operators. Assume now that the two input states $|\alpha\rangle$ and $|\alpha^*\rangle$ are sent each into one of the inputs of a balanced beam splitter, characterized by the canonical transformation

$$\hat{x}'_1 = (\hat{x}_1 + \hat{x}_2)/\sqrt{2}, \qquad \hat{p}'_1 = (\hat{p}_1 + \hat{p}_2)/\sqrt{2}, \tag{14}$$

$$\hat{x}'_2 = (\hat{x}_1 - \hat{x}_2)/\sqrt{2}, \qquad \hat{p}'_2 = (\hat{p}_1 - \hat{p}_2)/\sqrt{2}. \tag{15}$$

The input product state can be reexpressed as

$$|\alpha\rangle \otimes |\alpha^*\rangle = \exp(i\sqrt{2}\, p\, \hat{x}'_2 - i\sqrt{2}\, x\, \hat{p}'_1)|0\rangle \tag{16}$$

implying that x and p can be measured *separately* here by applying homodyne detection on modes 1' and 2'. Indeed, a measurement of the first quadrature of mode 1' yields $\sqrt{2}\,x$, on average, while a measurement of the second quadrature of mode 2' yields $\sqrt{2}\,p$. These two measurements suffer each from an error of variance $\Delta x^2_{\rm vac} = 1/2$. Hence, the resulting error variance on x and p is reduced to $\Delta x^2_{\rm vac}/2 = 1/4$. In contrast, if we had the input product state $|\alpha\rangle^{\otimes 2}$ and were sending each coherent state $|\alpha\rangle$ into an input of a balanced beam splitter, we would obtain a single coherent state $|\sqrt{2}\ \alpha\rangle$ on output mode 1'. One should then necessarily perform a *simultaneous* measurement of the two quadratures of the latter mode, yielding $(\sqrt{2}\,x, \sqrt{2}\,p)$ with an error variance $2\Delta x^2_{\rm vac} = 1$, or, equivalently x and p with a variance $\Delta x^2_{\rm vac} = 1/2$. As a consequence, we have proven here that a better strategy for sending x and p to Bob is to encode them into two conjugate coherent states $|(x+ip)/\sqrt{2}\rangle \otimes |(x-ip)/\sqrt{2}\rangle$ rather than sending two replicas of $|(x+ip)/\sqrt{2}\rangle$. The error variance on x and p is indeed reduced by a factor of two via the use of phase conjugation.

Finally, let us discuss the connection between the universal phase conjugator and quantum cloning. It can be shown that the Gaussian cloning machine for continuous variables introduced in [8] generates, in addition to the two clones of the input state, an imperfect phase-conjugate version of the input state with the same fidelity ($F = 1/2$) as that of the universal phase-conjugator [9]. The exact same property holds for the universal qubit cloner [10], which also yields a flipped qubit with a fidelity equal to that of the universal quantum spin-flip machine [2]. Thus, the general rule seems to apply that the production of two clones is necessarily accompanied by the creation of one anticlone (time-reversed state).

As a last comment, it is worthwhile noting that we have here another example of the classical nature of the universal-NOT operation. As emphasized in [2], spin flipping is essentially a classical operation on qubits, since it can be done by a measurement followed by the preparation of a flipped spin. This also implies that any number of flipped spins can be produced together with the same fidelity. Similarly, we have shown here that the same situation prevails for the phase conjugation of continuous quantum variables. It seems therefore tempting to conjecture that any (imperfect) time-reversal procedure can be done optimally in a classical way. Proving this conjecture and understanding the fundamental reason for it are interesting open questions.

We are grateful to Serge Massar for useful discussions. S. I. acknowledges support from the Fondation Universitaire Van Buuren at the Université Libre de Bruxelles.

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